Chapter 4

Space of Continuous Functions

Two fundamental results concerning the space of continuous functions are present. In Section 1 we characterize precompact sets in the space of continuous functions, and, as an application, Cauchy-Peano Theorem on the existence of the initial value problem for differential equations is derived. In Section 2 the notions of first and second category are introduced and then Baire Category Theorem is proved. As an application, it is shown that there are many continuous, but nowhere differentiable functions.

4.1 Ascoli's Theorem

Recall that for any metric space (X, d), the space of all bounded, continuous functions $C_b(X)$ forms a complete metric space under the supnorm. In this section we will consider X being $\overline{G}$, where is G be a bounded, open set in $\mathbb{R}^n$. Note that every continuous function in $\overline{G}$ is uniformly continuous and, in particular, bounded. This space shares many common features with its one-dimensional special case $C[a, b]$.

Unlike the Euclidean space, $C(\overline{G})$ is an infinite dimensional vector space. A classical result is Bolzano-Weierstrass Theorem asserting that every sequence in a bounded set in the Euclidean space contains a convergent subsequence. This property no longer holds for our space. Examples are bountiful, for instance, consider the unit ball in $C[0, 1]$ which is clearly bounded. It contains the sequence $\{f_n\},\ f_n(x) = x^n,\ x \in [0, 1]$ whose pointwise limit exists and is equal to the function $f(x) = 0,\ x \neq 1$ and $f(1) = 1$. Since f is not continuous, this sequence cannot have any convergent sequence in supnorm. Bolzano-Weierstrass Theorem does not hold in $C[0, 1]$.

To proceed further, let us introduce some basic definitions. A set E in a metric space is called a **precompact** set if every sequence E contains a convergent subsequence with limit in X. It is further called **compact** if the limit belongs to E. In other words, a compact set is a precompact set which is also closed. Indeed, let $\{x_n\} \subset E$ where E is precompact, some $\{x_{n_k}\} \to z$. As E is also closed, $z \in E$, so E is compact. By Bolzano-Weierstrass Theorem a set in $\mathbb{R}^n$ is precompact if and only if it is bounded, and a set is compact if and only if it is closed and bounded. On the other hand, as we have seen, only boundedness is not sufficient to ensure precompactness for sets in $C(\overline{G})$. In this section we characterize these sets.

The crux is the notion of equicontinuity. A set $\mathcal{C}$ in $C(\overline{G})$ is **equicontinuous** if for every $\varepsilon > 0$, there exists some δ such that

$$|f(x) - f(y)| < \varepsilon, \quad \text{for all } f \in \mathcal{C}, \quad \text{and} \quad |x - y| < \delta, \ x, y \in \overline{G}.$$

Recall that a function is uniformly continuous in $\overline{G}$ if for each $\varepsilon > 0$, there exists some δ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta, \ x, y \in \overline{G}$. Equicontinuity means that δ can further be chosen to fit all functions in $\mathcal{C}$. Applying to any single function in the set, it implies that each element in $\mathcal{C}$ is uniformly continuous.

There are various ways to show that a set is equicontinuous. Recall that a function f defined in a subset $\overline{G}$ of $\mathbb{R}^n$ is called Hölder continuous if there exists some $\alpha \in (0, 1)$ such that

$$|f(x) - f(y)| \le L|x - y|^\alpha, \quad \text{for all } x, y \in \overline{G}, \tag{4.1}$$

for some constant L. The number α is called the Hölder exponent. The function is called Lipschitz continuous if (4.1) holds for α equals to 1. The set $\mathcal{C}$ in $C(\overline{G})$ is said to satisfy a **uniform Hölder** or **Lipschitz condition** if all f's are Hölder continuous with the same α and L or Lipschitz continuous and (4.1) holds for the same constant L. Clearly, such a set is equicontinuous. In fact, for any $\varepsilon > 0$, any δ satisfying $L\delta^\alpha < \varepsilon$ can do the job. The following situation is commonly encountered in the study of differential equations. The philosophy is that equicontinuity can be obtained if there is a good, uniform control on the derivatives of the functions.

Proposition 4.1. *Let $\mathcal{C}$ be in $C(\overline{G})$ where $\overline{G}$ is convex in $\mathbb{R}^n$. Suppose that each $f \in \mathcal{C}$ is differentiable and there is a uniform bound on their partial derivatives. Then $\mathcal{C}$ is equicontinuous.*

Proof. We repeat an old argument. For, x and y in X, $(1-t)x + ty, \ t \in [0, 1]$, belongs to $\overline{G}$ by convexity. Let $\psi(t) \equiv f((1-t)x + ty), \ f \in \mathcal{C}$. By the chain rule

$$\psi'(t) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}((1-t)x + ty)(y_j - x_j),$$

we have

$$\begin{aligned} f(y) - f(x) &= \psi(1) - \psi(0) \\ &= \int_0^1 \psi'(t)dt \\ &= \sum_{j=1}^n \int_0^1 \frac{\partial f}{\partial x_j}(x + t(y-x))(y_j - x_j). \end{aligned}$$

Therefore,

$$|f(y) - f(x)| \le \sqrt{n} M |y - x|,$$

where $M = \sup\{|\partial f/\partial x_j(x)| : \ x \in X, \ j = 1, \ldots, n,\}$ after using Cauchy-Schwarz Inequality. We conclude that $\mathcal{C}$ satisfies a uniform Lipschitz condition with Lipschitz constant $n^{1/2}M$.

□

Example 4.1. Let

$$\mathcal{E} = \{x : \ x'(t) = t, \ t \in [-1, 1]\} \subset C[-1, 1].$$

As $|x(t) - x(s)| \le \|x'\|_\infty |t-s| \le |t-s|$, every function in $\mathcal{E}$ is Lipschitz continuous with Lipschitz constant equals to 1, $\mathcal{E}$ is equicontinuous. However, the functions $x_n(t) = t^2/2 + n$ belongs to $\mathcal{E}$ for all $n \ge 1$ and $x_n(0) = n \to \infty$ as $n \to \infty$. It shows that $\mathcal{E}$ is unbounded and cannot have any convergent subsequence. It shows that the boundedness of $\mathcal{E}$ cannot be removed in Ascoli's Theorem.

Example 4.2. Let

$$\mathcal{B} = \{f \in C[0, 1] : |f(x)| \le 1, x \in [0, 1]\} \subset C[0, 1].$$

Clearly $\mathcal{B}$ is closed and bounded. However, we do not have any uniform control on the oscillation of the functions in this set, so it should not be equicontinuous. In fact, consider the sequence $\{\sin nx\}, n \ge 1$, in $\mathcal{B}$. We claim that it is not equicontinuous. In fact, suppose for $\varepsilon = 1/2$, there exists some δ such that $|\sin nx - \sin ny| < 1/2$, whenever $|x - y| < \delta$ for all n. Pick a large n such that $n\delta > \pi$. Taking $x = 0$ and $y = \pi/2n$, $|x - y| < \delta$ but $|\sin nx - \sin ny| = |\sin \pi/2| = 1 > 1/2$, contradiction holds. Hence $\mathcal{B}$ is not equicontinuous.

In fact, $\mathcal{B}$ is the unit ball around 0 in $C[0, 1]$. A general result in functional analysis asserts that the unit ball of a normed space is precompact if and only if the space has finite dimension.

Theorem 4.2 (Ascoli's Theorem). *Consider $C(\overline{G})$ where G is bounded, open in $\mathbb{R}^n$. A set $\mathcal{E}$ in $C(\overline{G})$ is precompact if it is bounded and equicontinuous.*

We need the following useful lemma from elementary analysis.

Lemma 4.3. *Let A be a countable set and $\{f_n\}$ be a sequence of real-valued functions defined on A. Suppose that for each $z \in A$, there exists an M such that $|f_n(z)| \le M$ for all $n \ge 1$. There is a subsequence of $\{f_n\}$, $\{f_{n_k}\}$, such that $\{f_{n_k}(z)\}$ is convergent at each $z \in A$.*

Proof. Let $A = \{z_j\}, j \ge 1$. Since $\{f_n(z_1)\}$ is a bounded sequence, we can extract a subsequence $\{f_n^1\}$ such that $\{f_n^1(z_1)\}$ is convergent. Next, as $\{f_n^1(z_2)\}$ is bounded, it has a subsequence $\{f_n^2\}$ such that $\{f_n^2(z_2)\}$ is convergent. Keep doing in this way, we obtain sequences $\{f_n^j\}$ satisfying (i) $\{f_n^{j+1}\}$ is a subsequence of $\{f_n^j\}$ and (ii) $\{f_n^j(z_1)\}, \{f_n^j(z_2)\}, \cdots, \{f_n^j(z_j)\}$ are convergent. Then the diagonal sequence $\{g_n\},\ g_n = f_n^n,$ for all $n \ge 1$, is a subsequence of $\{f_n\}$ which converges at every z_j. $\square$

The subsequence selected in this way is usually called a Cantor's diagonal sequence. It first came up in Cantor's study of infinite sets.

Proof of Ascoli's Theorem. Let $\{f_n\}$ be a sequence in $\mathcal{E}$. For each $k \ge 0$, consider the set

$$E_k \equiv \{x = (x_1, \cdots, x_n) \in \overline{G} : \ x_j = \frac{m}{2^k},\ m \in \mathbb{Z},\ j = 1, \cdots, n\} ,$$

and

$$E = \bigcup_k E_k \ .$$

Each E_k is a finite set and hence E is a countable set $\{z_j\}$. By assumption, there is some M satisfying $|f_n(x)| \le M$ for all $x \in \overline{G}$. Using the lemma above, we can pick a subsequence from $\{f_n\}$, $\{g_n\}$, such that $\{g_n(z_j)\}$ is convergent for each z_j's. We claim that $\{g_n\}$ is a Cauchy sequence in $C(\overline{G})$. For, by equicontinuity, for every $\varepsilon > 0$, there exists a δ such that $|g_n(x) - g_n(y)| < \dfrac{\varepsilon}{3}$, whenever $|x - y| < \delta,\ x, y \in \overline{G}$. We fix k so that $1/2^k < \delta$. Observing that the union of all ball of radius $1/2^k$ centered at points in each E_k covers $\overline{G}$. Therefore, for each $x \in \overline{G}$, we can find some $z_j \in E_k$ such that $|x - z_j| < \delta$. We have

$$\begin{aligned} |g_n(x) - g_m(x)| &\le |g_n(x) - g_n(z_j)| + |g_n(z_j) - g_m(z_j)| + |g_m(z_j) - g_m(x)| \\ &< \frac{\varepsilon}{3} + |g_n(z_j) - g_m(z_j)| + \frac{\varepsilon}{3}. \end{aligned}$$

As $\{g_n(z_j)\}$ converges, there exists n_0 (depending on z_j) such that

$$|g_n(z_j) - g_m(z_j)| < \frac{\varepsilon}{3}, \quad \text{for all } n, m \ge n_0. \tag{4.2}$$

As E_k is a finite set, we can choose some N_0 such that (4.2) holds for all z_j and $n, m \ge N_0$. It follows that

$$|g_n(x) - g_m(x)| < \varepsilon, \quad \text{for all } n, m \ge N_0,$$

i.e., $\{g_n\}$ is a Cauchy sequence in $C(\overline{G})$. By the completeness of $C(\overline{G})$, $\{g_n\}$ converges uniformly to some function in $C(\overline{G})$. We have completed the proof of Ascoli's Theorem.

Example 4.3. This example shows that the underlying space G must be bounded. Let φ be a non-zero, C^1-function in $[0,1]$ which vanishes outside $[1/2, 3/4]$. Define a sequence $\{f_n\}$ in $C[0,\infty)$ by setting $f_n(x) = \varphi(x-n), x \in [n, n+1]$ and $= 0$ elsewhere. Then $\|f_n\|_\infty = \|\varphi\|_\infty > 0$ is bounded and clearly it is also equicontinuous. But it has no convergent subsequence. Why? Suppose some $\{f_{n_j}\}$ converges to f in supnorm (that is, uniformly convergent). It must also be pointwisely convergent, $f_{n_j}(x) \to f(x)$ for all x. As $f_n(x) = 0$ for all $n \geq x+1$, $f(x) \equiv 0$. But then, $\|\varphi\|_\infty = \|f_{n_j}\|_\infty = \|f_{n_j} - f\|_\infty \to 0$, which is impossible.

On the other hand, based on the fact that any uniformly continuous function defined on a bounded, open set G can be extended to be a uniformly continuous function in $\overline{G}$, one can show that Theorem 4.2 remains valid if we assume $\mathcal{E}$ is bounded and equicontinuous in $C(G)$. We leave the details to you.

The following result provides a converse to Ascoli's theorem.

Theorem 4.4 (Arzela's Theorem). *Every precompact set in $C(\overline{G})$ must be bounded and equicontinuous.*

Proof. Let $\mathcal{E}$ be precompact. If $\mathcal{E}$ is unbounded, there exists some sequence $\{f_n\}$ in $\mathcal{E}$ satisfying $\sup_n \|f_n\| = \infty$. Pick a subsequence $\{f_{n_j}\}$ satisfying $\|f_{n_j}\| \to \infty$ as $n_j \to \infty$. Clearly this subsequence cannot contain any convergent subsequence.

Next, assume that $\mathcal{E}$ is bounded and precompact but not equicontinuous. We are going to draw a contradiction. First of all, there is some $\varepsilon_0 > 0$ such that, for each $n \geq 1$, we can find $f_n,\ x_n, y_n \in \overline{G}$ satisfying

$$|f_n(x_n) - f_n(y_n)| \geq \varepsilon_0, \quad |x_n - y_n| < \frac{1}{n} \ .$$

By precompactness, $\{f_n\}$ contains a convergent subsequence, let it be $\{f_{n_j}\}$. Assume that $f_{n_j} \rightrightarrows f \in C(\overline{G})$ as $n_j \to \infty$. The sequence $\{x_{n_j}\} \subset \overline{G}$ contains a convergent subsequence $x_{n_{j_k}} \to z \in \overline{G}$. To simplify notation, we let $g_k = f_{n_{j_k}}$ and $x'_k = x_{n_{j_k}}, y'_k = y_{n_{j_k}}$. For $\varepsilon > 0$, there is some n_0 such that

$$\begin{aligned} |g_k(x'_k) - g_k(y'_k)| &\leq |g_k(x'_k) - f(x'_k)| + |f(x'_k) - f(y'_k)| + |f(y'_k) - g_k(y'_k)| \\ &\leq \|g_k - f\|_\infty + |f(x'_k) - f(y'_k)| + \|g_k - f\|_\infty \\ &< \varepsilon + |f(x'_k) - f(y'_k)| + \varepsilon \ , \quad \forall n_{j_k} \geq n_0 \ . \end{aligned}$$

As $x'_k \to z$ and $|x'_k - y'_k| < 1/n_{j_k} \to 0$, $y'_k \to z$ too. Therefore, there exists some n_1 such that $|f(x'_k) - f(y'_k)| \leq |f(x'_k) - f(z))| + |f(z) - f(y'_k)| < \varepsilon$, for all $n \geq n_1$. It follows that

$$|g_k(x'_k) - g_k(y'_k)| < 3\varepsilon \ ,$$

for all $n_{j_k} \geq \max\{n_0, n_1\}$. We get a contradiction when $\varepsilon < \varepsilon_0/3$. Hence, $\mathcal{E}$ cannot be equicontinuous. $\square$

We present an application of Ascoli's Theorem to ordinary differential equations. Consider the initial value problem again,

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \\ \\ x(t_0) = x_0. \end{cases} \tag{IVP}$$

where f is a continuous function defined in the rectangle $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$. In Chapter 3 we proved that this Cauchy problem has a *unique* solution when f satisfies the Lipschitz condition. Now we show that the existence part of Picard-Lindelöf Theoorem is still valid without the Lipschitiz condition.

We start with an improvement on the domain of existence for the unique solution in Picard-Lindelöf Theorem. Recall that it was shown the solution exists on the interval $(t_0 - a', t_0 + a')$ where

$$a' = \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\} \ .$$

Now we have

Proposition 4.5. *Under the setting of Picard-Lindelöf Theorem, the unique solution exists on the interval* $(t_0 - a^*, t_0 + a^*)$ *where*

$$a^* = \min\left\{a, \frac{b}{M}\right\} \ .$$

Proof. Let $w_+(t) = M(t - t_0) + x_0$ and $w_-(t) = -M(t - t_0) + x_0$ where $M = \sup_R |f|$ as before. From $|x'(t)| \leq M$ and $x(t_0) = x_0$ it is easy to see that the solution of (IVP) satisfies $w_-(t) \leq x(t) \leq w_+(t)$ as long as it exists. Let us assume $b/M \leq a$ so that $a^* = b/M$ (the other case can be handled in the same way). The straight lines $x = M(t - t_0) + x_0$ and $x = -M(t - t_0) + x_0, t \geq t_0$, intersects $x = x_0 + b$ and $x = x_0 - b$ respectively at two points P and Q. The graph of $x(t)$ stays inside the triangle with vertices at $(t_0, x_0), P$ and Q for $t \geq t_0$. We are going to show it exists on $[t_0, t_0 + a^*)$. Let

$$\alpha^* = \sup\{\alpha \leq a^* : \text{ the solution exists on } (t_0, t_0 + \alpha)\} > 0 \ .$$

(This set is nonempty because a' is there.) Assume on the contrary $\alpha^* < a^*$. The vertical line $x = \alpha^*$ intersects $x = M(t - t_0) + x_0$ and $x = -M(t - t_0) + x_0$ respectively at P' and Q'.

The triangle Δ with vertices at (t_0, x_0), P' and Q' is a compact set contained in the interior of R. Therefore, $\rho = d(\Delta, \partial R) > 0$. We can find a square $S = [-s, s] \times [-s, s], s = \rho/2$, so that $S_{x,t} \equiv S + (t, x)$ is contained inside R for every $(t, x) \in \Delta$. By applying Picard-Lindelöf Theorem to the equation with initial point (t, x) we conclude that there is a unique solution of the equation over some interval with center at t whose length l is independent of (x, t). Now, since our solution $x(t)$ exists on every closed subinterval of $[0, \alpha^*)$, we can fix some $t_1, \alpha^* - t_1 < l/2$. We solve the equation using $(t_1, x(t_1))$ as our initial point to get a solution x_1 defined on an interval of length at least l around t_1. But, using the uniqueness property, the solution $x(t)$ and $x_1(t)$ together formed a solution whose domain extends beyond α^*, contradiction holds. We conclude that the solution exists up to $t_0 + a^*$, that is $\alpha^* = a^*$. Similarly, we could show that the solution exists on $(t_0 - a^*, t_0]$.

□

Theorem 4.6 (**Cauchy-Peano Theorem**). *Consider (IVP) where f is continuous on $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$. There exist $a' \in (0, a)$ and a C^1-function $x : [t_0 - a', t_0 + a'] \to [x_0 - b, x_0 + b]$, solving (IVP).*

From the proof we will see that a' can be taken to be any number in $(0, \min\{a, b/M\})$ where $M = \sup\{|f(t, x)| : (t, x) \in R\}$. The theorem is also valid for systems.

Proof. By Weierstrass Approximation Theorem, there exists a sequence of polynomials $\{p_n\}$ approaching f in $C(R)$ uniformly. In particular, we have $M_n \to M$, where

$$M_n = \max\{|p_n(t, x)| : (t, x) \in R\}.$$

As each p_n satisfies the Lipschitz condition, there is a unique solution x_n defined on $I_n = (t_0 - a_n, t_0 + a_n), a_n = \min\{a, b/M_n\}$ for the initial value problem

$$\frac{dx}{dt} = p_n(t, x), \quad x_0(t_0) = x_0.$$

(The Lipschitz constants may depend on p_n, but this does no harm.) As $a_n \to a^* \equiv \min\{a, b/M\}$, the domain of existence I_n eventually expands to $(t_0 - a^*, t_0 + a^*)$ as $n \to \infty$.

Let $a' < a^*$ be fixed. There exists some n_0 such that x_n is well-defined on $[t_0 - a', t_0 + a']$ for all $n \geq n_0$. From $|dx_n/dt| \leq M_n$ and $\lim_{n\to\infty} M_n = M$, we can fix some $n_1 \geq n_0$ such that $M_n \leq M + 1$ for all $n \geq n_1$. It follows that $\{x_n\}$ forms an equicontinuous set over $[t_0 - a', t_0 + a']$. On the other hand, from

$$x_n(t) = x_0 + \int_{t_0}^{t} p_n(s, x_n(s))ds, \tag{4.3}$$

we have

$$|x_n(t)| \leq |x_0| + aM_n \leq |x_0| + a(M + 1), \quad n \geq n_1.$$

It follows that $\{x_n\}, n \geq n_1$, is bounded on $[t_0-a', t_0+a']$. By Ascoli's Theorem, it contains a subsequence $\{x_{n_j}\}$ converging uniformly to a continuous function x^* on $[t_0 - a', t_0 + a']$. It remains to check that x^* solves (IVP) on this interval. Passing limit in (4.3), clearly its left hand side tends to $x^*(t)$. To treat its right hand side, we observe that, for $\varepsilon > 0$, there exists some δ such that

$$|f(s_2, x_2) - f(s_1, x_1)| < \varepsilon, \quad |s_2 - s_1|, |x_2 - x_1| < \delta \ . \tag{4.4}$$

This is because f is uniformly continuous on R. Next, there is some $n_2 \geq n_1$ such that $\|f - p_n\|_\infty < \varepsilon$, for all $n \geq n_2$ on R. It implies

$$|p_n(s, x) - f(s, x)| < \varepsilon, \quad \forall (s, x) \in R \ . \tag{4.5}$$

Putting (4.4) and (4.5) together, we have

$$\begin{aligned} & \left| \int_{t_0}^t p_{n_j}(s, x_{n_j}(s))ds - \int_{t_0}^t f(s, x^*(s))ds \right| \\ \leq & \left| \int_{t_0}^t |p_{n_j}(s, x_{n_j}(s)) - f(s, x_{n_j}(s))|ds \right| + \left| \int_{t_0}^t |f(s, x_{n_j}(s)) - f(s, x^*(s))|ds \right| \\ \leq & \ 2a\varepsilon \ , \end{aligned}$$

which implies the right hand side of (4.3) tends to

$$x_0 + \int_{t_0}^t f(s, x^*(s))ds \ ,$$

as $n_j \to \infty$. We conclude that x^* is a solution to (IVP) on $[t_0 - a', t_0 + a']$. □

We conclude this section with an example of non-uniqueness. Consider the function $x(t) = t^2/4$. A direct check shows that it satisfies the differential equation $x' = x^{1/2}$ for $t \geq 0$. Therefore, the IVP

$$x' = |x|^{1/2} \ , \quad x(0) = 0 \ ,$$

has two solutions. First, $x_1(t) \equiv 0$ is the trivial solution. On the other hand, $x_2(t) = t^2/4, t \geq 0$, and $= 0,\ t < 0$ is also a solution. In fact, there are two more solutions, namely, $x_3(t) = 0, t \geq 0, = -t^2/4, t < 0$ and $x_4 = x_2 + x_3$. Note that here $f(t, x) = |x|^{1/2}$ does not satisfy the Lipschitz condition in any interval containing the origin.

4.2 Baire Category Theorem

In this section we discuss Baire category theorem, a basic property of complete metric spaces. It is concerned with the decomposition of a metric space into a countable union of subsets. Although this theorem applies to all complete metric spaces, it has wonderful applications when it comes to the space of continuous functions. In view of this, we include the theorem here.

We start with a definition. A set E in a metric space (X, d) is a **dense set** if for every $x \in X$, the metric ball $B_r(x) \cap E \neq \phi$ for all $r > 0$. For instance, the set of all rational numbers $\mathbb{Q}$ forms a dense set in $\mathbb{R}$, so is the set of all irrational numbers $\mathbb{I}$. On the other hand, the set of all polynomials, $\mathcal{P}$ forms a dense set in $C[0, 1]$ due to Weierstrass Approximation Theorem.

A dense set is considered to be a "large set" in a metric space. However, this classification is a little too rough. Consider the following sets in $\mathbb{R}$ under the usual Euclidean metric:

$$\mathbb{R}, \quad \mathbb{R} \setminus \{a_1, \cdots, a_n\}, \quad \mathbb{I}, \quad \mathbb{Q} \ .$$

Although all of them are dense, they are quite different. Apparently the first two sets, which are open and dense, are "more dense" than the other two. On the other hand, $\mathbb{I}$ is "more dense" than $\mathbb{Q}$. How can we make it precise? A more satisfactory description of the size of a set, as we will see, can be achieved when working on a complete space.

Let us follow most books to start by looking from the other direction . A set E in a metric space is called **nowhere dense** if its closure has empty interior. A closed, nowhere dense set is precisely the complement of an open dense set. From the definition we see that the closure of a nowhere dense set is nowhere dense, and so are its subsets. A nowhere dense set is a set whose elements are scattered sparsely in the metric space. You may say a nowhere dense set is a small set.

Proposition 4.7. *(a) The closure of a nowhere dense set is nowhere dense, so are its subsets.*

(b) The union of finitely many nowhere dense sets is nowhere dense.

(c) Every finite set is nowhere dense provided the metric space has no isolated points.

A point a in a metric space is an **isolated point** if $\{a\}$ is an open set. As it is obvious that a singleton set is closed, an isolated point forms an open and closed set. Every point in a discrete metric space is isolated. There are no isolated points in $\mathbb{R}$. However, if we consider $\mathbb{Z}$ as a subspace of $\mathbb{R}$, every point is an isolated point. In fact, the metric ball $B_{1/2}(n)$ is equal to the singleton set $\{n\}$ showing that the latter is open.

Proof. (a) trivially follows from the definition of a nowhere dense set.

(b). A set is nowhere dense if and only if its closure is nowhere dense. Taking complement, it suffices to show that any finite intersection of open, dense sets is again dense. Furthermore, as any finite intersection of open sets is open, it reduces to showing the intersection of two open dense sets is dense. Let G_1 and G_2 be two open dense sets. For each $x \in X$ and $\varepsilon > 0$, consider the ball $B_\varepsilon(x)$. We need to show that $B_\varepsilon(x) \bigcap (G_1 \bigcap G_2)$ is non-empty. Indeed, since G_1 is dense, there exists some $x_1 \in G_1 \bigcap B_\varepsilon(x)$. Since G_1 is also open, so is $G_1 \bigcap B_\varepsilon(x)$. There is some small $\rho > 0$ such that $B_\rho(x_1) \subset G_1 \bigcap B_\varepsilon(x)$. As G_2 is dense, $B_\rho(x_1) \bigcap G_2$ is non-empty, so there is some $x_2 \in G_2 \bigcap B_\rho(x_1) \subset G_1 \bigcap G_2$, done.

(c). First of all, any finite set is closed. In view of (b), it suffices to show that the singleton set $\{a\}$ formed by an isolated point is nowhere dense, or, $X \setminus \{a\}$ is dense. But the latter is clearly true since every open set containing a must intersect $X \setminus \{a\}$. □

Example 4.4. Consider $\mathbb{R}$ under the Euclidean metric. As there are no isolated points, every finite set in $\mathbb{R}$ is nowhere dense. How about countably union of finite sets? The answer is sometimes yes and sometimes no. For instance the set $\{1, 2, 3, \cdots\}$ is closed and nowhere dense, but $\mathbb{Q}$, which is a countable set, is not nowhere dense.

Although countable unions of nowhere dense sets may not be nowhere dense, they are still considered by mathematicians to be small in size. We fix the notion by introducing the following definition. A set in a metric space is of **first category** or **meager** if it can be expressed as the countable union of nowhere dense sets. It is **of second category** if it is not meager. A set is called **residual** if its complement is of first category. As we will see, a residual set is most effective to describe a large set. First, we show that all nowhere dense sets are of first category.

Proposition 4.8. *(a) Every subset of a set of first category is of first category.*

(b) The union of countably many sets of first category is of first category.

(c) Every countable set is of first category provided the metric space has no isolated points.

Proof. (a) Let $E = \bigcup_n E_n$ be a set of first category where E_n's are nowhere dense. For $F \subset E$, $F = \bigcup_n (F \cap E_n)$ is of first category since $F \cap E_n$ is nowhere dense.

(b) It is clear from definition.

(c) It follows from Proposition 4.7.

□

Taking complement, Proposition 4.8 may be formulated in terms of residual sets.

Proposition 4.8'. (a) *Every set containing a residual set is residual.*

(b) *The intersection of countably many residual sets is a residual set.*

(c) *Every set whose complement is countable is a residual set provided the metric space has no isolated points.*

Example 4.5. Let $\mathbb{Q} = \{q_j\}, j \geq 1$ as a subset in $\mathbb{R}$. Since each singleton set is closed and nowhere dense, $\mathbb{Q}$, a countable union of rational numbers, is of first category. According to the definition, $\mathbb{I}$, the irrational numbers, is a residual set.

We agree that a nowhere dense set is regarded as a small set and so is its countable union. On the other hand, an open dense set is regarded as a large set and so is its countable intersection. This has been witnessed by the example $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$. Such observation can be justified when the ambient space is complete.

Theorem 4.9 (**Baire Category Theorem**). *In a complete metric space, the countable union of nowhere dense sets has empty interior.*

Proof. Let $F_n, n \geq 1$, be nowhere dense. Without loss of generality we may assume they are closed. Let B_0 be any ball. The theorem will be established if we can show that $B_0 \bigcap (X \setminus \bigcup_n F_n) \neq \phi$. As F_1 has empty interior, there exists some point $x_1 \in B_0$ lying outside F_1. Since F_1 is closed , we can find a closed ball $\overline{B}_1 \subset B_0$ centering at x_1 such that $\overline{B}_1 \cap F_1 = \phi$ and its diameter $d_1 \leq d_0/2$, where d_0 is the diameter of B_0. Next, as F_2 has empty interior and closed, by the same reason there is a closed ball $\overline{B}_2 \subset B_1$ centering at x_2 such that $\overline{B}_2 \cap F_2 = \phi$ and $d_2 \leq d_1/2$. Repeating this process, we obtain a sequence of closed balls $\overline{B}_n$ with center x_n satisfying (a) $\overline{B}_{n+1} \subset B_n$, (b) $d_n \leq d_0/2^n$, and (c) $\overline{B}_n$ is disjoint from $F_1, \cdots, F_n$. As the balls shrink to a point, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, $\{x_n\}$ converges to some x^*. As each $\overline{B_n}$ is closed and $x_j \in \overline{B_n}$ for all $j \geq n$, $x^* \in \overline{B_n}$ for all n. In particular, it means that x^* does not belong to F_n for all n, so x^* is a point in B_0 not in $\bigcup_n F_n$.

□

Using the fact that the complement of a closed, nowhere dense set is an open dense set, we have the following equivalent formulation of the Baire Category Theorem.

Theorem 5.9' (**Baire Category Theorem**). *Let (X, d) be a complete metric space and $\{G_n\}$ be a sequence of open, dense sets in X. The set $E = \bigcap_{n=1}^{\infty} G_n$ is dense.*

We have the following two immediate consequences of the theorem.

Corollary 4.10. *Let (X, d) be complete. Suppose*

$$X = \bigcup_{n=1}^{\infty} E_n \ ,$$

where E_n's are closed sets. Then at least one of these E_n's has non-empty interior.

In other words, it is impossible to decompose a complete metric space into a countable union of nowhere dense sets.

Proof. In case all these E_n's have empty interior, $X = \bigcup_{n=1}^{\infty} E_n$ has empty interior according to Baire Category Theorem. However, X is the entire space, so $X^o = X$ is non-empty, contradiction holds.

□

Corollary 4.11. *If a set in a complete metric space is of first category, it cannot be a residual set, and vice versa.*

Proof. Let E be of first category and let $E \subset \bigcup_{n=1}^{\infty} F_n$ where F_n's are closed with empty interior. If it is residaul, its complement is of first category. Thus, $X \setminus E = \bigcup_{n=1}^{\infty} E_n$ where E_n's are closed with empty interior. We put F_n's and E_n's together to form a sequence $\{H_n\}, H_{2n-1} = F_n, H_{2n} = E_n$. Then

$$X = E \bigcup (X \setminus E) \subset \bigcup_{n=1}^{\infty} H_n \ ,$$

which contradicts the corollary above. □

Baire Category Theorem has many interesting applications. We end this section by giving two standard ones. It is concerned with the existence of continuous, but nowhere differentiable functions. We knew that Weierstrass is the first person who constructed such a function in 1896. His function is explicitly given in the form of an infinite series

$$W(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}.$$

Here we use an implicit argument to show there are far more such functions than continuously differentiable functions.

To refresh your memory we state

Lemma 4.12. *Let $f \in C[a,b]$ be differentiable at x. Then it is Lipschitz continuous at x.*

Proof. By differentiability, for $\varepsilon = 1$, there exists some δ_0 such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < 1, \quad \forall y \neq x, |y - x| < \delta_0.$$

We have

$$|f(y) - f(x)| \leq L|y - x|, \quad \forall y, |y - x| < \delta_0,$$

where $L = |f'(x)| + 1$. For y lying outside $(x - \delta_0, x + \delta_0)$, $|y - x| \geq \delta_0$. Hence

$$\begin{aligned} |f(y) - f(x)| &\leq |f(x)| + |f(y)| \\ &\leq \frac{2M}{\delta_0}|y - x|, \quad \forall y \in [a,b] \setminus (x - \delta_0, x + \delta_0), \end{aligned}$$

where $M = \sup\{|f(x)| : x \in [a,b]\}$. It follows that f is Lipschitz continuous at x with an Lipschitz constant not exceeding $\max\{L, 2M/\delta_0\}$.

□

Theorem 4.13. *The set of all continuous, nowhere differentiable functions forms a residual set in $C[a,b]$ and hence dense in $C[a,b]$.*

Proof. For each $L > 0$, define

$$S_L = \big\{f \in C[a,b] : f \text{ is Lipschitz continuous at some } x \text{ with the Lipschitz constant } \leq L\big\}.$$

We claim that S_L is a closed set. For, let $\{f_n\}$ be a sequence in S_L which is Lipschitz continuous at x_n and converges uniformly to f. We need to show $f \in S_L$. By passing to a subsequence if necessary, we may assume $\{x_n\}$ to some x^* in $[a,b]$. We have, by letting $n \to \infty$,

$$\begin{aligned}
|f(y) - f(x^*)| &\leq |f(y) - f_n(y)| + |f_n(y) - f(x^*)| \\
&\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x_n)| + |f_n(x_n) - f_n(x^*)| + |f_n(x^*) - f(x^*)| \\
&\leq |f(y) - f_n(y)| + L|y - x_n| + L|x_n - x^*| + |f_n(x^*) - f(x^*)| \\
&\to L|y - x^*| \ .
\end{aligned}$$

Next we show that each S_L is nowhere dense. Let $f \in S_L$. By Weierstrass Approximation Theorem, for every $\varepsilon > 0$, we can find some polynomial p such that $\|f - p\|_\infty < \varepsilon/2$. Since every polynomial is Lipschitz continuous, let the Lipschitz constant of p be L_1. Consider the function $g(x) = p(x) + (\varepsilon/2)\varphi(x)$ where φ is the jig-saw function of period $2r$ satisfying $0 \leq \varphi \leq 1$ and $\varphi(0) = 1$. The slope of this function is either $1/r$ or $-1/r$. Both will become large when r is chosen to be small. Clearly, we have $\|f - g\|_\infty < \varepsilon$. On the other hand,

$$\begin{aligned}
|g(y) - g(x)| &\geq \frac{\varepsilon}{2}|\varphi(y) - \varphi(x)| - |p(y) - p(x)| \\
&\geq \frac{\varepsilon}{2}|\varphi(y) - \varphi(x)| - L_1|y - x| \ .
\end{aligned}$$

For each x sitting in $[a,b]$, we can always find some y nearby so that the slope of φ over the line segment between x and y is greater than $1/r$ or less than $-1/r$. Therefore, if we choose r so that

$$\frac{\varepsilon}{2}\frac{1}{r} - L_1 > L,$$

we have $|g(y) - g(x)| > L|y - x|$, that is, g does not belong to S_L.

Denoting by S the set of functions in $C[a,b]$ which are differentiable at at least one point, by the lemma it must belong to S_N for some positive integer N. Therefore, $S \subset \cup_{N=1}^\infty S_N$ is of first category. Its complement, the collection of continuous, nowhere differentiable functions, is residual, and hence dense in $C[0,1]$. The completeness of $C[0,1]$ is implicitly used. □

Though elegant, a drawback of this proof is that one cannot assert which particular function is nowhere differentiable. On the other hand, the example of Weierstrass is a concrete one.

Our second application is concerned with the basis of a vector space. Recall that a basis of a vector space is a set of linearly independent vectors such that every vector can be expressed as a linear combination of vectors from the basis. More precisely, Let V be a real vector space. A subset B of V is called a basis of V if (a) Let $w_1, w_2, \cdots, w_N$ be finitely many vectors from B. If $a_1 w_2 + a_2 w_2 + \cdots + a_N w_N = 0$, then $a_1 = a_2 = \cdots = a_n = 0$, that is, these w_j's are linearly independent; and (b) every vector from V can be written as $v = \sum_{j=1}^M a_j w_j$ for some finite number M, that is, every vector can be expressed as a linear combination of vectors from B. The construction of a basis in a finite dimensional vector space was done in MATH2040. However, in an infinite dimensional vector space the construction of a basis is not so easy. Nevertheless, using Zorn's lemma, a variant of the axiom of choice, one shows that a basis always exists. Some authors call a basis for an infinite dimensional basis a Hamel basis. The difficulty in writing down a Hamel basis is explained in the following result.

Theorem 4.14. *Every basis of an infinite dimensional Banach space consists of uncountably many vectors.*

Proof. Let V be an infinite dimensional Banach space and $\mathcal{B} = \{w_j\}$ be a countable basis. We aim for a contradiction. Indeed, let W_n be the subspace spanned by $\{w_1, \cdots, w_n\}$. We have the decomposition

$$V = \bigcup_n W_n.$$

If one can show that each W_n is closed and has empty interior, since V is complete, the corollary above tells us this decomposition is impossible. To see that W_n has empty interior, pick a unit vector v_0 outside W_n. For $w \in W_n$ and $\varepsilon > 0$, $w + \varepsilon v_0 \in B_\varepsilon(w) \cap (V \setminus W_n)$, so W_n cannot contain a ball. Next, letting v_j be a sequence in W_n and $v_j \to v_0$, we would like to show that $v \in W_n$. Indeed, every vector $v \in W_n$ can be uniquely expressed as $\sum_{j=1}^n a_j w_j$. The map $v \mapsto a \equiv (a_1, \cdots, a_n)$ sets up a linear bijection between W_n and $\mathbb{R}^n$ and $|||a||| \equiv \|v\|$ defines a norm on $\mathbb{R}^n$. Since any two norms in $\mathbb{R}^n$ are equivalent (see exercise), a convergent (resp. Cauchy) sequence in one norm is the same in the other norm. Since now $\{v_j\}$ is convergent in V, it is a Cauchy sequence in V. The corresponding sequence $\{a^j\}$, $a^j = (a_1^j, \cdots, a_n^j)$, is a Cauchy sequence in $\mathbb{R}^n$ with respect to $|||\cdot|||$ and hence in $\|\cdot\|_2$, the Euclidean norm. Using the completeness of $\mathbb{R}^n$ with respect to the Euclidean norm, $\{a^j\}$ converges to some $a^* = (a_1^*, \cdots, a_n^*)$. But then $\{v_j\}$ converges to $v^* = \sum_j a_j^* w_j$ in W_n. By the uniqueness of limit, we conclude that $v_0 = v^* \in W_n$, so W_n is closed.

□

Comments on Chapter 4.

Arzela and Ascoli Theorems play the role in the space of continuous functions the same as Bolzano-Weierstrass theorem does in the Euclidean space. A bounded sequence of real numbers always admits a convergent subsequence. Although this is no longer true for bounded sequences of continuous functions on $[a, b]$, it does hold when the sequence is also equicontinuous. A more general version of Ascoli's Theorem asserts that Theorem 4.2 still holds when $C(\overline{G})$ is replaced by $C(X)$ where X is a compact metric space. Google for more.

Ascoli's Theorem (Theorem 4.2) is widely applied in the theory of partial differential equations, the calculus of variations, complex analysis and differential geometry. Here is a taste of how it works for a minimization problem. Consider

$$\inf \left\{ J[u] : \ u(0) = 0, u(1) = 5, \ u \in C^1[0,1] \right\},$$

where

$$J[u] = \int_0^1 \Big(u'^2(x) - \cos u(x) \Big) \, dx.$$

First of all, we observe that $J[u] \geq -1$. This is clear, for the cosine function is always bounded by ± 1. After knowing that this problem is bounded from -1, we see that $\inf J[u]$ must be a finite number, say, γ. Next we pick a minimizing sequence $\{u_n\}$, that is, every u_n is in $C^1[0,1]$ and satisfies $u(0) = 0, u(1) = 5$, such that $J[u_n] \to \gamma$ as $n \to \infty$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \big| u_n(x) - u_n(y) \big| &\leq \int_x^y \big| u_n'(x) \big| dx \\ &\leq \sqrt{\int_x^y 1^2 dx} \sqrt{\int_x^y u_n'^2(x) dx} \\ &\leq \sqrt{\int_x^y 1^2 dx} \sqrt{\int_0^1 u_n'^2(x) dx} \\ &\leq \sqrt{J[u_n] + 1} \sqrt{|y - x|} \\ &\leq \sqrt{\gamma + 2} \ |y - x|^{1/2} \end{aligned}$$

for all large n. From this estimate we immediately see that $\{u_n\}$ is equicontinuous and bounded (because $u_n(0) = 0$). By Ascoli's Theorem, it has a subsequence $\{u_{n_j}\}$ converging to some $u \in C[0,1]$. Apparently, $u(0) = 0, u(1) = 5$. Using knowledge from functional analysis, one can further argue that $u \in C^1[0,1]$ and is the minimum of this problem.

There is an alternate proof of Cauchy-Peano Theorem without using Picard-Lindelöf Theorem. In this proof piecewise linear approximate solutions, called Euler's polygonal lines, to (IVP) are constructed and subconvergence to a solution is shown by Ascoli's theorem, see, Coddington-Levinson, Theory of Ordinary Differential Equations, for details.

There are some fundamental results that require completeness. The contraction mapping principle is one and Baire category theorem is another. The latter was first introduced by Baire in his 1899 doctoral thesis. It has wide, and very often amazing applications in all branches of analysis. Some nice applications are available on the web. Google under applications of Baire category theorem for more.

Weierstrass' example is discussed in Hewitt-Stromberg, "Abstract Analysis". An simpler example can be found in Rudin's Principles.

Being unable to locate a single reference containing these three topics, I decide not to name any reference but let you search through the internet.